

Highly structured tensor identities for (2,2)-forms in four dimensions.

Ola Wingbrant*

February 7, 2008

Abstract

In an n dimensional vector space, any tensor which is antisymmetric in $k > n$ arguments must vanish; this is a trivial consequence of the limited number of dimensions. However, when other possible properties of tensors, for example trace-freeness, are taken into account, such identities may be heavily disguised. Tensor identities of this kind were first considered by Lovelock, and later by Edgar and Höglund.

In this paper we continue their work. We obtain dimensionally dependent identities for highly structured expressions of products of (2,2)-forms. For tensors possessing more symmetries, such as block symmetry $W_{abcd} = W_{cdab}$, or the first Bianchi identity $W_{a[bcd]} = 0$, we derive identities for less structured expressions.

These identities are important tools when studying super-energy tensors, and, in turn, deriving identities for them. As an application we are able to show that the Bel-Robinson tensor, the super-energy tensor for the Weyl tensor, satisfies the equation $\mathcal{T}_{abcy}\mathcal{T}^{abcx} = \frac{1}{4}\delta_y^x\mathcal{T}_{abcd}\mathcal{T}^{abcd}$ in four dimensions, irrespective of the signature of the space.

PACS: 04.20.Cv

1 Introduction

The natural language for general relativity in four dimensions, and with a space having Lorentzian signature, is spinors [13]. The dimension and signature is built into the spinor formalism, and this is both its strength and its weakness. Many important relations are obvious or easy to derive in spinors, while their tensor counterparts are more difficult.

However, if we want to go beyond four dimensions and/or use metrics with signatures other than Lorentzian, the tensor formalism is the usual choice at hand. So in these spaces other means to obtain such important relations have to be used. One method for deriving tensor relations that has proven useful is to exploit so called “dimensionally dependent identities” [3, 7, 11, 12].

This method is built around the process of antisymmetrizing over more arguments than the number of dimensions of the space. It was first considered by David Lovelock in the late 1960’s [11, 12], when he drew attention to a set of

*Valhallagatan 26B SE-582 43 Linköping
E-mail: ola@wingbrant.info

apparently unrelated tensor identities used in general relativity. He managed to show that they were all special cases of two basic identities, both having a very simple structure. Lovelock gave two theorems, and showed that they were consequences of the dimension alone.

Lovelock's work has recently resurfaced and his theorems have been generalized by Edgar and Höglund [7]. We will continue their work, and show how their ideas can be used in proving a set of relations for highly structured tensor expressions.

Some of the four-dimensional results presented here are already known in the special case of a space with Lorentz signature, and are more or less trivial in the spinor formalism. However, by considering such relations in tensors instead, we are guided in the derivation of the corresponding tensor relations. This knowledge may then be used when trying to prove, or refute, the existence of analogous tensor relations in higher dimensions [8].

A wide class of tensors that have been shown to be significant in both four and higher dimensions are the super-energy tensors; see [14] and references therein. One example is the Bel-Robinson tensor [1], which was originally constructed in four dimensions as the super-energy tensor for the gravitational field. However, it has subsequently been used in, e.g., eleven dimensional supergravity theories [6].

In addition, Bergqvist [2] has shown that, in four dimensions, certain properties of super-energy tensors are investigated more easily using the equivalent super-energy spinors.

As an application of the identities we derive, we show how to obtain a quadratic identity for the Bel-Robinson tensor,

$$\mathcal{T}_{abcy}\mathcal{T}^{abcx} = \frac{1}{4}\delta_y^x\mathcal{T}_{abcd}\mathcal{T}^{abcd} \quad (1)$$

by purely tensorial means.

In this paper we will make use of the abstract index notation as defined in [13]. We also introduce the convention that T is any tensor, without any specified trace or symmetry properties. U will always symbolize a **trace-free (2,2)-form**. That is,

$$U^a{}_{bad} = 0 \quad U_{abcd} = U_{[ab]cd} = U_{ab[cd]}. \quad (2)$$

V will be taken to be a **block symmetric, trace-free (2,2)-form**, hence

$$V^a{}_{bad} = 0 \quad V_{abcd} = V_{[ab]cd} = V_{ab[cd]} \quad V_{abcd} = V_{cdab}. \quad (3)$$

Finally, W will stand for a **Weyl candidate**. A Weyl candidate is a four index tensor with all the algebraic properties of the Weyl tensor, and thus

$$W^a{}_{bad} = 0 \quad W_{abcd} = W_{[ab]cd} = W_{ab[cd]} \quad W_{abcd} = W_{cdab} \quad W_{a[bcd]} = 0. \quad (4)$$

The last relation of (4) is often referred to as the **first Bianchi identity**. Note that nothing is said about the differential properties of W . Note also that each class is a special case of the one before, and hence, results obtained for U will apply to both V and W etc.

Their counterparts without the trace-free property will be represented by U , V and W respectively.

2 Basic quadratic identities

In this section we will follow [13] and adopt the convention that $T^{\mathcal{A}}_{a_1}$ denotes a tensor which, in addition to the free index a_1 , has an arbitrary free index configuration of upper and/or lower indices \mathcal{A} .

In an n -dimensional vector space, we have

$$T^{\mathcal{A}}_{[a_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \dots \delta_{a_{n+1}}^{b_{n+1}}] = 0, \quad (5)$$

since the left hand side is antisymmetric in more indices than the dimension of the space. By taking more indices explicitly into the antisymmetrization we are able to obtain other identities, which contain fewer deltas,

$$T^{\mathcal{A}}_{[a_1 \dots a_k} \delta_{a_{k+1}}^{b_{k+1}} \delta_{a_{k+2}}^{b_{k+2}} \dots \delta_{a_{n+1}}^{b_{n+1}}] = 0. \quad (6)$$

Such identities would only involve the antisymmetric part of T with respect to the explicit indices. However, when T itself is antisymmetric in these indices, equation (6) will give us an identity with fewer deltas than (5), but without losing any part of T .

The arguments work equally well for (k, l) -forms. Hence,

$$T^{\mathcal{A}}_{a_1 \dots a_k} [b_1 \dots b_l \delta_{a_{k+1}}^{b_{k+1}} \delta_{a_{k+2}}^{b_{k+2}} \dots \delta_{a_{n+1}}^{b_{n+1}}] = 0 \quad (7)$$

when $l \geq k$.

These ideas have been thoroughly investigated, and the next theorem is due to [7].

Theorem 2.1 *In an n -dimensional space let $T^{\mathcal{A}}_{a_1 \dots a_k} b_1 \dots b_l = T^{\mathcal{A}}_{[a_1 \dots a_k]} [b_1 \dots b_l]$ be trace-free on its explicit indices. Then*

$$T^{\mathcal{A}}_{[a_1 \dots a_k} [b_1 \dots b_l \delta_{a_{k+1}}^{b_{l+1}} \dots \delta_{a_{k+d}}^{b_{l+d}}] = 0 \quad (8)$$

where $d + k + l \geq n + 1$ and $d \geq 0$. \diamond

Proof Antisymmetrizing over $k + l + d \geq n + 1$ indices, we have

$$0 = T^{\mathcal{A}}_{[a_1 \dots a_k} i_1 \dots i_l \delta_{a_{k+1}}^{b_{l+1}} \dots \delta_{a_{k+d}}^{b_{l+d}} \delta_{i_1}^{b_1} \dots \delta_{i_l}^{b_l}] \quad (9)$$

$$= T^{\mathcal{A}}_{[a_1 \dots a_k} i_1 \dots i_l \delta_{a_{k+1}}^{[b_{l+1}} \dots \delta_{a_{k+d}}^{b_{l+d}} \delta_{i_1}^{b_1} \dots \delta_{i_l}^{b_l]} \quad (10)$$

Since T is trace-free on explicit indices, this reduces to

$$0 = T^{\mathcal{A}}_{[a_1 \dots a_k} i_1 \dots i_l \delta_{a_{k+1}}^{[b_{l+1}} \dots \delta_{a_{k+d}}^{b_{l+d}} \delta_{i_1}^{b_1} \dots \delta_{i_l}^{b_l]} \quad (11)$$

The theorem follows by absorbing the deltas. \diamond

Since we will restrict our discussions to $(2, 2)$ -forms in four dimensions, Theorem 2.1 will not be exploited to its full extent. In this special case, we have

Corollary 2.2 *Let U^{ab}_{cd} be a trace-free $(2, 2)$ -form. In four dimensions,*

$$U^{[ab}_{[cd} \delta_{y]}^{x]} = 0. \quad (12)$$

\diamond

We will now derive three identities, which we will use frequently throughout the rest of this paper. They are all simple but elegant consequences of Corollary 2.2. The most compact presentation of each of these identities is for trace-free forms; but as will be seen, alternative presentations are possible without the trace-free condition.

Our first result is one of the identities considered by Lovelock [12]. We give the proof in detail since it illustrates the basic idea.

Theorem 2.3 *Let $U^{ab}{}_{cd}$ be a trace-free (2,2)-form. In four dimensions*

$$U^{xb}{}_{cd}U^{cd}{}_{yb} = \frac{1}{4}\delta_y^x U^{ab}{}_{cd}U^{cd}{}_{ab} \quad (13)$$

◇

Proof By Corollary 2.2 we have

$$U^{[ab}{}_{[cd}\delta_{y]}^x] = 0. \quad (14)$$

Expanding the left hand side gives 36 terms. Using the antisymmetries of U , this is reduced by a factor 4 to 9 terms:

$$\begin{aligned} 0 = & U^{xb}{}_{yd}\delta_c^a + U^{xb}{}_{cy}\delta_d^a + U^{ax}{}_{yd}\delta_c^b + U^{ax}{}_{cy}\delta_d^b - \\ & -U^{xb}{}_{cd}\delta_y^a - U^{ax}{}_{cd}\delta_y^b - U^{ab}{}_{yd}\delta_c^x - U^{ab}{}_{cy}\delta_d^x + \\ & +U^{ab}{}_{cd}\delta_y^x. \end{aligned} \quad (15)$$

Multiplication by $U^{cd}{}_{ab}$ and using the trace-free property yields

$$-4U^{xb}{}_{cd}U^{cd}{}_{yb} + \delta_y^x U^{ab}{}_{cd}U^{cd}{}_{ab} = 0. \quad (16)$$

◇

Note that if equation (14) is multiplied by $U_{ab}{}^{cd}$ instead, we get

$$U^{xb}{}_{cd}U_{yb}{}^{cd} = \frac{1}{4}\delta_y^x U^{ab}{}_{cd}U_{ab}{}^{cd}. \quad (17)$$

If U_{abcd} is also block symmetric, then (17) will of course be equal to (13).

Now, suppose that U_{abcd} is the trace-free part of some (2,2)-form \mathbb{U}_{abcd} , and defined by the equation

$$U_{abcd} = \mathbb{U}_{abcd} - \frac{2}{(n-2)}(g_{a[c}\mathbb{U}_{d]b} - g_{b[c}\mathbb{U}_{d]a}) + \frac{2}{(n-2)(n-1)}\mathbb{U}g_{a[c}g_{d]b} \quad (18)$$

where $\mathbb{U}_{ab} = \mathbb{U}^k{}_{akb}$ and $\mathbb{U} = \mathbb{U}^k{}_k$. We may then substitute U_{abcd} for (18) in (13), and we have

Corollary 2.4 *In four dimensions*

$$\begin{aligned} & U^{xb}{}_{cd}U^{cd}{}_{yb} + \mathbb{U}\mathbb{U}^x{}_y - 2\mathbb{U}^x{}_c\mathbb{U}^c{}_y - 2U^{xb}{}_{yd}\mathbb{U}^d{}_b = \\ & \frac{1}{4}\delta_y^x (\mathbb{U}^{ab}{}_{cd}U^{cd}{}_{ab} + \mathbb{U}\mathbb{U} - 4\mathbb{U}^b{}_d\mathbb{U}^d{}_b) \end{aligned} \quad (19)$$

for any (2,2)-form $\mathbb{U}^{ab}{}_{cd}$.

◇

If in addition, the tensor is block symmetric, we are able to derive

Theorem 2.5 *Let V_{abcd} be a trace-free, block symmetric (2,2)-form. In four dimensions*

$$V^x{}_{bcd}V^{bcd}{}_y = \frac{1}{4}\delta_y^x V^a{}_{bcd}V^{bcd}{}_a \quad (20)$$

◇

The proof is completely analogous to that of Theorem 2.3 and therefore omitted. The result may also be found with slightly more work by considering the identity

$$V^{[a}{}_{[bcd}\delta_{y]}^x]V_a{}^{bcd} = 0, \quad (21)$$

which holds by Theorem 2.1.

In the same way as for Theorem 2.3 we may think of V_{abcd} as the trace-free part of a block symmetric (2,2)-form V_{abcd} . We have

Corollary 2.6 *For any block symmetric (2,2)-form V_{abcd} ,*

$$\begin{aligned} V_{abc}{}^x V^{acb}{}_y - V^x{}_a V^a{}_y - V^{xc}{}_{yb} V^b{}_c + \frac{1}{2}V^x{}_y V = \\ \frac{1}{4}\delta_y^x \left(V_{abc}{}^d V^{acb}{}_d - 2V^c{}_b V^b{}_c + \frac{1}{2}V V \right) \end{aligned} \quad (22)$$

◇

This relation will prove to be useful in Section 6.

The next lemma is a bit different; it is concerned with an expression with four free indices.

Lemma 2.7 (Index switch) *Let U_{abcd} be a trace-free (2,2)-form. In four dimensions*

$$2U^{b[w}{}_{cy}U^{c]z}{}_b = \frac{1}{2}U^{wx}{}_{bc}U^{bc}{}_{yz} - \frac{1}{4}\delta_y^{[w}\delta_z^{x]}U^{ab}{}_{cd}U^{cd}{}_{ab} \quad (23)$$

◇

Proof Consider the identity

$$U^{[wb}{}_{[cd}\delta_{y]}^x]U^{cd}{}_{zb} = 0. \quad (24)$$

Expansion and use of equation (13) gives

$$\begin{aligned} 0 = & -\frac{1}{4}\delta_y^x \delta_z^w U^{ab}{}_{cd}U^{cd}{}_{ab} + \frac{1}{4}\delta_y^w \delta_z^x U^{ab}{}_{cd}U^{cd}{}_{ab} - \\ & -U^{wx}{}_{cd}U^{cd}{}_{yz} + 2U^{bw}{}_{cy}U^{xc}{}_{zb} - 2U^{bx}{}_{cy}U^{wc}{}_{zb}, \end{aligned} \quad (25)$$

which is (23). ◇

Note that relation (23) may be put in the form

$$U^{bw}{}_{cy}U^{xc}{}_{zb} = U^{bx}{}_{cy}U^{wc}{}_{zb} + \frac{1}{2}U^{wx}{}_{cd}U^{cd}{}_{yz} - \frac{1}{4}\delta_y^{[w}\delta_z^{x]}U^{ab}{}_{cd}U^{cd}{}_{ab}. \quad (26)$$

Hence it may be interpreted as a switch of the abstract indices w and x .

This lemma will be an invaluable tool in proving Lemma 6.2, which in turn, is essential in showing one of our main results, Theorem 7.2.

3 Chains of trace-free (2,2)-forms

There are some very highly structured product constructions for (2,2)-forms. We will call them **chains** because of the way the tensors are linked to each other. We will call the number of participating tensors in a chain the **chains length**.

Definition 3.1 (Chain of the zeroth kind) *An expression of the form*

$$\underbrace{T^{wx}_{c_1 d_1} T^{c_1 d_1}_{c_2 d_2} T^{c_2 d_2}_{c_3 d_3} \cdots T^{c_{m-2} d_{m-2}}_{c_{m-1} d_{m-1}} T^{c_{m-1} d_{m-1}}_{yz}}_m \quad (27)$$

where indices w, x, y and z are free, is said to be a **chain of the zeroth kind** of length m and is written $T_0^m[m]^{wx}_{yz}$. \diamond

Theorem 3.2 *Let U^{ab}_{cd} be a trace-free (2,2)-form. In four dimensions*

$$U_0^m[m]^{xb}_{yb} = \frac{1}{4} \delta_y^x U_0^m[m]^{ab}_{ab} \quad (28)$$

\diamond

Proof The proof is by induction over the number of factors. We know that the case $m = 2$ is true by equation (13). Assume that (28) holds for $m = k - 1 \geq 2$. Consider the identity

$$U^{ab}_{[cd} \delta_y^x] U_0^{[k-1]cd}_{ab} = 0. \quad (29)$$

Expansion gives,

$$\begin{aligned} 0 &= U^{ab}_{cd} \delta_y^x U_0^{[k-1]cd}_{ab} + \\ &\quad + (U^{xb}_{yd} \delta_c^a + U^{xb}_{cy} \delta_d^a + U^{ax}_{yd} \delta_c^b + U^{ax}_{cy} \delta_d^b) U_0^{[k-1]cd}_{ab} + \\ &\quad + (-U^{xb}_{cd} \delta_y^a - U^{ax}_{cd} \delta_y^b - U^{ab}_{yd} \delta_c^x - U^{ab}_{cy} \delta_d^x) U_0^{[k-1]cd}_{ab} \end{aligned} \quad (30)$$

After renaming dummy indices we are left with

$$\delta_y^x U^{ab}_{cd} U_0^{[k-1]cd}_{ab} = 4U^{xb}_{cd} U_0^{[k-1]cd}_{yb} - 4U^{xb}_{yd} U_0^{[k-1]cd}_{cb} \quad (31)$$

However, the last term of equation (31) vanishes since,

$$U^{xb}_{yd} U_0^{[k-1]cd}_{cb} = U^{xb}_{yd} \delta_b^d U_0^{[k-1]ca}_{ca} = 0 \quad (32)$$

The first equality of (32) holds by the induction hypothesis, and the second one follows from the trace-free property. Thus we are left with

$$\delta_y^x U^{ab}_{cd} U_0^{[k-1]cd}_{ab} = 4U^{xb}_{cd} U_0^{[k-1]cd}_{yb} \quad (33)$$

which is (28). \diamond

Definition 3.3 (Chain of the first kind) *An expression of the form*

$$\underbrace{T^{wc_1}_{yd_1} T^{d_1 c_2}_{c_1 d_2} T^{d_2 c_3}_{c_2 d_3} \cdots T^{d_{m-2} c_{m-1}}_{c_{m-2} d_{m-1}} T^{d_{m-1} x}_{c_{m-1} z}}_m \quad (34)$$

where indices w, x, y and z are free, is said to be a **chain of the first kind** of length m , and is written $T[m]^{wx}_{yz}$. \diamond

Note that when T is a $(2, 2)$ -form U , we see directly from the definition that

$$U[m]^{wx}_{yz} = U[m]^{xw}_{zy} \quad (35)$$

Theorem 3.4 *Let U^{ab}_{cd} be a trace-free $(2, 2)$ -form. In four dimensions*

$$U[m]^{xb}_{by} = \frac{1}{4} \delta_y^x U[m]^{ab}_{ba} \quad (36)$$

\diamond

Proof The proof is similar to the proof of Theorem 3.2 with only slight modifications. From (13) we know that (36) holds in the case $m = 2$. Assume that it holds for $m = k - 1 \geq 2$, and consider the identity

$$U^{[ab}_{[cd} \delta_y^x] U[m-1]^{dc}_{ba} = 0. \quad (37)$$

Expanding this relation gives

$$\begin{aligned} 0 &= U^{ab}_{cd} \delta_y^x U[k-1]^{dc}_{ba} + \\ &\quad (U^{xb}_{yd} \delta_c^a + U^{xb}_{cy} \delta_d^a + U^{ax}_{yd} \delta_c^b + U^{ax}_{cy} \delta_d^b) U[k-1]^{dc}_{ba} + \\ &\quad + (-U^{xb}_{cd} \delta_y^a - U^{ax}_{cd} \delta_y^b - U^{ab}_{yd} \delta_c^x - U^{ab}_{cy} \delta_d^x) U[k-1]^{dc}_{ba} \end{aligned} \quad (38)$$

We see directly that two of the terms on the second row disappear because of the trace-free property. If we rewrite the rest of the terms, using property (35) and renaming dummy indices, we have

$$\delta_y^x U^{ab}_{cd} U[k-1]^{dc}_{ba} = 4U^{xb}_{cd} U[k-1]^{dc}_{by} - 2U^{xb}_{cy} U[k-1]^{cd}_{db} \quad (39)$$

The last term of equation (39) vanishes since,

$$U^{xb}_{cy} U[k-1]^{cd}_{db} = U^{xb}_{cy} \delta_b^c U[k-1]^{ad}_{da} = 0 \quad (40)$$

The first equality of (40) holds by the induction hypothesis, the second one follows from the trace-free property. Therefore we are left with

$$\delta_y^x U^{ab}_{cd} U[k-1]^{dc}_{ba} = 4U^{xb}_{cd} U[k-1]^{dc}_{by} \quad (41)$$

and the proof is finished. \diamond

Next we are concerned with one kind of broken chain structure. It is a chain of the zeroth kind, but it includes an element that destroys the ordinary regularity.

Theorem 3.5 *In four dimensions, with $k > 0$ and $l > 0$,*

$$U[k]^{xb}_{cd} U^{de}_{bf} U[m]^{fc}_{ey} = \frac{1}{4} \delta_y^x U[k]^{ab}_{cd} U^{de}_{bf} U[m]^{fc}_{ea} \quad (42)$$

for all trace-free $(2, 2)$ -forms U . \diamond

Proof Expansion of the identity

$$U_0[k]^{xf}_{ab} U^{[ab]}_{[cd]f} \delta_f^e U[m]^{cd}_{ye} = 0 \quad (43)$$

yields

$$\begin{aligned} 0 = & U_0[k+m+1]^{xf}_{yf} - 2U_0[k]^{xf}_{bf} U_0[m+1]^{be}_{ye} - \\ & - 2U_0[k+1]^{xf}_{bf} U_0[m]^{be}_{ye} - 4U_0[k]^{xf}_{ab} U^{be}_{fd} U_0[m]^{da}_{ey} \end{aligned} \quad (44)$$

The first three terms are chains of the zeroth kind, and products of chains of the zeroth kind. Thus, by Theorem 3.2 we have

$$\begin{aligned} U_0[k]^{xf}_{ab} U^{be}_{fd} U_0[m]^{da}_{ey} = & \frac{1}{16} \delta_y^x U_0[k+m+1]^{af}_{af} - \\ & - \frac{1}{32} \delta_y^x U_0[k]^{af}_{af} U_0[m+1]^{de}_{de} - \frac{1}{32} \delta_y^x U_0[k+1]^{af}_{af} U_0[m]^{de}_{de} \end{aligned} \quad (45)$$

Taking the trace of equation (45) and multiplication by $\frac{1}{4} \delta_y^x$ gives

$$\begin{aligned} \frac{1}{4} \delta_y^x U_0[k]^{cf}_{ab} U^{be}_{fd} U_0[m]^{da}_{ec} = & \frac{1}{16} \delta_y^x U_0[k+m+1]^{af}_{af} - \\ & - \frac{1}{32} \delta_y^x U_0[k]^{af}_{af} U_0[m+1]^{de}_{de} - \frac{1}{32} \delta_y^x U_0[k+1]^{af}_{af} U_0[m]^{de}_{de} \end{aligned} \quad (46)$$

and the theorem follows. \diamond

The proof immediately gives us a relation between this more complicated expression and chains of the zeroth kind.

Corollary 3.6 *In four dimensions*

$$\begin{aligned} U_0[k]^{xf}_{ab} U^{be}_{fd} U_0[m]^{da}_{ey} = & \frac{1}{16} \delta_y^x \left(U_0[k+m+1]^{af}_{af} - \right. \\ & \left. - \frac{1}{2} U_0[k]^{af}_{af} U_0[m+1]^{de}_{de} - \frac{1}{2} U_0[k+1]^{af}_{af} U_0[m]^{de}_{de} \right) \end{aligned} \quad (47)$$

\diamond

In Section 6 we will explicitly use the special case of Theorem 3.5 when $k = 1$ and $m = 2$. We therefore give this result as a separate corollary.

Corollary 3.7 *In four dimensions*

$$U^{xb}_{cd} U^{de}_{bf} U^{cf}_{gh} U^{gh}_{ye} = \frac{1}{4} \delta_y^x U^{ab}_{cd} U^{de}_{bf} U^{cf}_{gh} U^{gh}_{ae} \quad (48)$$

\diamond

4 Chains of trace-free block symmetric (2,2)-forms

What happens if the structures of the previous section are disturbed in some other way? Is it still possible to find identities? In order to investigate these

questions further we start by looking at a chain of the first kind. For simplicity we will carry out our arguments in the case when the chain is of length five, although the results are valid for chains of arbitrary length.

$$T^{wb}_{yd}T^{de}_{bf}T^{fg}_{eh}T^{hi}_{gk}T^{kx}_{iz} \quad (49)$$

Let g and h change places in the third T , we then have

$$T^{wb}_{yd}T^{de}_{bf}T^{fh}_{he}T^{hi}_{gk}T^{kx}_{iz} \quad (50)$$

We shall call such a construction a **twist**.

From now on, until the end of this section, we will only discuss trace-free (2,2)-forms enjoying the block symmetric property $V_{abcd} = V_{cdab}$. Since this class of tensors is a special case of those of the Section 3, any V_{abcd} will, of course, obey the results of that section. However, we are able to derive additional results, originating from the block symmetry.

We immediately see that

$$V^{wb}_{yd}V^{de}_{bf}V^{fh}_{he}V^{hi}_{gk}V^{kx}_{iz} = V^{wb}_{yd}V^{de}_{bf}V^{fh}_{eg}V^{gi}_{kh}V^{kx}_{iz}. \quad (51)$$

Thus, twists may move freely along chains, and we may always assume that the twist is at the end of the chain. We shall write such a chain of length m as $V^{wx}_{yz}_{t1}$. A chain of length m that contains $n < m$ twists, is denoted by $V^{wx}_{yz}_{nt1}$.

Assume that we have a chain with two twists. Because of what we stated above we can further assume that they are on neighboring tensors.

$$V^{wb}_{yd}V^{de}_{bf}V^{fh}_{he}V^{hi}_{kg}V^{kx}_{iz} = V^{wb}_{yd}V^{de}_{bf}V^{fh}_{eg}V^{gi}_{hk}V^{kx}_{iz} \quad (52)$$

By the block symmetry it is clear that the twists cancel. Hence, we have the following lemma.

Lemma 4.1 (Twist reduction) *Let V_{abcd} be a trace-free, block symmetric (2,2)-form. Then*

$$V^{wx}_{yz}_{nt1} = V^{wx}_{yz}_{t1} \quad (53)$$

if the number of twists, n , are odd, and

$$V^{wx}_{yz}_{nt1} = V^{wx}_{yz}_1 \quad (54)$$

if the number of twists are even. \diamond

Hence, the number of twists reduce to either zero or one, depending on whether it is even or odd to start with.

Theorem 4.2 *In four dimensions,*

$$V^{xb}_{by}_{t1} = \frac{1}{4}\delta^x_y V^{ab}_{ba}_{t1} \quad (55)$$

for any trace-free block symmetric (2,2)-form V . \diamond

Proof The proof is analogous to those of Theorem 3.2 and Theorem 3.4, and we will therefore leave out some of the details.

Equation (55) is true for $m = 2$ by Theorem 2.5. Assume it is true for $m = k - 1 \geq 2$, and consider the identity

$$V^{[ab}_{[cd}\delta^x_{y]}V[k-2]^{di}_{bk}V^k_{ai}{}^c = 0 \quad (56)$$

Expansion and use of symmetries, trace properties, and induction hypothesis yields

$$0 = V_1[k-1]^{ai}_{ck}V^k_{ai}{}^c\delta^x_y - 4V^{xb}_{cd}V_1[k-2]^{di}_{bk}V^k_{yi}{}^c, \quad (57)$$

which is the desired result. \diamond

5 Chains of Weyl candidates.

As we pointed out at the beginning of Section 2, Weyl candidates are special cases of block symmetric (2,2)-forms, and will therefore obey all the results of both Section 3 and Section 4. But, since Weyl candidates also possess the first Bianchi identity, $W_{a[bcd]} = 0$, we are able to derive an additional result. In order to do so, we turn our attention to a variant of chains of the zeroth kind.

Let us, in analogy with the previous section, study a chain of the zeroth kind of length five.

$$T^{wx}_{cd}T^{cd}_{ef}T^{ef}_{gh}T^{gh}_{ik}T^{ik}_{yz} \quad (58)$$

Now, twist f and h in the middle factor,

$$T^{wx}_{cd}T^{cd}_{ef}T^{e\ f}_{hg}T^{gh}_{ik}T^{ik}_{yz} \quad (59)$$

We note that this may be rewritten as,

$$\begin{aligned} T^{wx}_{cd}T^{cd}_{ef}T^{e\ f}_{hg}T^{gh}_{ik}T^{ik}_{yz} = \\ -T^{wx}_{cd}T^{cd}_{ef}T^{e\ f}_{hg}T^{hg}_{ik}T^{ik}_{yz} = T^{wx}_{cd}T^{cd}_{ef}T^{e\ f}_{gh}T^{gh}_{ik}T^{ik}_{yz} \end{aligned} \quad (60)$$

Evidently, twists gives rise to elements that break the structure in this case as well. Chains of the zeroth kind containing such elements are denoted by $T_{t0}^{[m]wx}_{yz}$.

Lemma 5.1 *For any Weyl candidate W_{abcd}*

$$W_a{}^e{}_b{}^f W_{ef}{}^{cd} = \frac{1}{2}W_{ab}{}^{ef}W_{ef}{}^{cd} \quad (61)$$

\diamond

Proof By the first Bianchi identity we have

$$W_{ab}{}^{ef} = (W_a{}^e{}_b{}^f - W_a{}^f{}_b{}^e) = 2W_a{}^{[e}{}_b{}^{f]}. \quad (62)$$

And hence

$$W_a{}^e{}_b{}^f W_{ef}{}^{cd} = W_a{}^e{}_b{}^f W_{[ef]}{}^{cd} = W_a{}^{[e}{}_b{}^{f]} W_{ef}{}^{cd} = \frac{1}{2}W_{ab}{}^{ef}W_{ef}{}^{cd} \quad (63)$$

\diamond

Theorem 5.2 *For chains of the zeroth kind of any Weyl candidate W_{abcd} of length m , with $n < m$ number of twists*

$$W_{nt0}[m]^{wx}_{yz} = \frac{1}{2^n} W_0[m]^{wx}_{yz}. \quad (64)$$

◇

The proof is immediate from Lemma 5.1.

Corollary 5.3

$$W_{nt0}[m]^{xb}_{yb} = \frac{1}{4 \cdot 2^n} \delta_y^x W_0[m]^{ab}_{ab} \quad (65)$$

for chains of Weyl candidates W_{abcd} with $n < m$ number of twists. ◇

6 Quartic identities

In this section we will break the chain structures of Section 3 even further. However, we will specialize our discussion to chains of length four.

Consider an expression of a block symmetric double 2-form

$$V^{xc}_{ef} V^{ef}_{ab} V^a_{cgh} V^{gh}_y{}^b. \quad (66)$$

This would be a chain of the zeroth kind if b and c changed places in the last two factors. In order to make the next theorem slightly more general, we will rearrange this expression. Using the block symmetries, we have

$$V^{xc}_{ef} V^{ef}_{ab} V^a_{cgh} V^{gh}_y{}^b = V^{xc}_{ef} V_{ab}{}^{ef} V^a_{cgh} V_y{}^{bgh} \quad (67)$$

Lemma 6.1 *In four dimensions*

$$U^{xc}_{ef} U_{ab}{}^{ef} U^a_{cgh} U_y{}^{bgh} = \frac{1}{4} \delta_y^x U^{dc}_{ef} U_{ab}{}^{ef} U^a_{cgh} U_d{}^{bgh} \quad (68)$$

for all trace-free $(2,2)$ -forms U_{abcd} . ◇

Proof The main idea is to let a product of two trace-free $(2,2)$ -forms substitute for V in equation (22). Put

$$V_{abcd} = U_{ab}{}^{ef} U_{cdef} \quad (69)$$

Since U is a $(2,2)$ -form so is V . In addition V has the block symmetry as well. For the trace of V we have by equation (17)

$$V_a{}^b{}_{bd} = U_a{}^{bef} U_{bdef} = \frac{1}{4} g_{ad} U^{cbef} U_{bcef} = \frac{1}{4} g_{ad} V^{cb}{}_{bc} \quad (70)$$

We may now consider relation (22). However, by virtue of (70) this simplifies to

$$0 = -V_{abc}{}^d V^{acb}{}_d \delta_y^x + 4 V_{abc}{}^x V^{acb}{}_y \quad (71)$$

or equivalently

$$V^x{}_{cab} V^{ac}{}_y{}^b = \frac{1}{4} \delta_y^x V^d{}_{cab} V^{ac}{}_d{}^b \quad (72)$$

Hence,

$$U^x_{cef} U_{ab}^{ef} U_{gh}^{ac} U_y^{bgh} = \frac{1}{4} \delta_y^x U_{cef}^d U_{ab}^{ef} U_{gh}^{ac} U_d^{bgh} \quad (73)$$

as required. \diamond

For completeness we note that the following substitutions are also possible in (22). They are all block symmetric and the trace has the property (70).

$$V^{ab}_{cd} = V^{i[a}_{j[c} V^{b]j}_{d]i} \quad (74)$$

$$V^{ab}_{cd} = V_j^{[a|i} [c V^{b]j}_{d]i} \quad (75)$$

$$V^{ab}_{cd} = V_{ij}^{[a} [c V^{b]ij}_{d]} \quad (76)$$

$$V^{ab}_{cd} = U_i^{[a} [c U^{b]i}_{j d]} \quad (77)$$

The first two yield identical expansions; they may with some effort, and together with Lemma 2.7, be used to derive Lemma 6.1. The last two merely reproduce Lemma 6.1 in a more indirect way. Hence nothing new can be found from any of these.

Lemma 6.2 *Let W_{abcd} be a Weyl candidate. In four dimensions*

$$W^{xb}_{cd} W^{de}_{bf} W^{fgc}_h W^h_{egy} = \frac{1}{4} \delta_y^x W^{ab}_{cd} W^{de}_{bf} W^{fgc}_h W^h_{ega} \quad (78)$$

\diamond

Proof We note that expression (78) would be a chain of the first kind of length four if c and e in the last two factors were interchanged. Start with the left-hand-side,

$$W^{xb}_{cd} W^{de}_{bf} \underbrace{W^{fgc}_h W^h_{egy}}_{\text{Bianchi}} = \quad (79)$$

$$= -W^{xb}_{cd} W^{de}_{bf} \underbrace{W^{fc}_h W^{gh}_{egy}}_{\text{lem. 5.1}} - W^{xb}_{cd} W^{de}_{bf} \underbrace{W^{fgc}_h W^h_{egy}}_{\text{lem. 2.7, } c \leftrightarrow e} = \quad (80)$$

$$= -\frac{1}{2} W^{xb}_{cd} W^{de}_{bf} W^{fc}_{gh} W^{gh}_{ey} + \quad (81)$$

$$+ W^{xb}_{cd} W^{d}_{eb} W^{f} \left(W^{ge}_{hf} W^{ch}_{yg} + \frac{1}{2} W^{ce}_{gh} W^{gh}_{fy} - \frac{1}{4} \delta_f^{[c} \delta_y^{e]} W^{ij}_{kl} W^{kl}_{ij} \right) =$$

$$= -\frac{1}{2} W^{xb}_{cd} W^{de}_{bf} W^{fc}_{gh} W^{gh}_{ey} + W^{xb}_{cd} W^{d}_{eb} W^{f} W^{ge}_{hf} W^{ch}_{yg} + \quad (82)$$

$$+ \frac{1}{2} W^{xb}_{cd} \underbrace{W^{d}_{eb} W^{f} W^{ce}_{gh} W^{gh}_{fy}}_{\text{Bianchi}} - \frac{1}{8} W^{x}_{bcd} W^{bcd}_y W^{ij}_{kl} W^{kl}_{ij} =$$

$$= -\frac{1}{2} W^{xb}_{cd} W^{de}_{bf} W^{fc}_{gh} W^{gh}_{ey} + W^{xb}_{cd} W^{d}_{eb} W^{f} W^{eg}_{fh} W^{hc}_{gy} - \quad (83)$$

$$- \frac{1}{2} \underbrace{W^{xb}_{cd} W^{d}_{bf} W^{f} W^{ce}_{gh} W^{gh}_{fy}}_{\text{lem. 5.1}} - \frac{1}{2} W^{xb}_{cd} W^{d}_{eb} W^{f} W^{ce}_{gh} W^{gh}_{fy} -$$

$$- \frac{1}{8} W^{x}_{bcd} W^{bcd}_y W^{ij}_{kl} W^{kl}_{ij} = \quad (84)$$

$$\begin{aligned}
&= - \underbrace{W^{xb}_{cd} W^{de}_{bf} W^{fc}_{gh} W^{gh}_{ey}}_{\text{cor. 3.7}} + \underbrace{W^{xb}_{cd} W^d_{eb} W^{ef}_{fh} W^{hc}_{gy}}_{\text{th. 4.2}} \quad (85) \\
&\quad + \frac{1}{4} \underbrace{W^x_{cbd} W^{bdf}_e W^{ce}_{gh} W^{gh}_{fy}}_{\text{lem. 6.1}} - \frac{1}{8} \underbrace{W^x_{bcd} W^{bcd}_y}_{\text{th. 4.2}} W^{ij}_{kl} W^{kl}_{ij} = \\
&= - \frac{1}{4} \delta_y^x W^{ab}_{cd} W^{de}_{bf} W^{fc}_{gh} W^{gh}_{ea} + \frac{1}{4} \delta_y^x W^{ab}_{cd} W^d_{eb} W^{ef}_{fh} W^{hc}_{ga} \quad (86) \\
&\quad + \frac{1}{16} \delta_y^x W^a_{cbd} W^{bdf}_e W^{ce}_{gh} W^{gh}_{fa} - \frac{1}{32} \delta_y^x W^a_{bcd} W^{bcd}_a W^{ij}_{kl} W^{kl}_{ij}
\end{aligned}$$

Taking the trace of (86) and multiplying by $\frac{1}{4}\delta_y^x$ gives the lemma. \diamond

It is worth noting that this result may also be found in a slightly different way. By using only the block- and antisymmetries of a Weyl candidate, together with Lemma 2.7 several times, we have

$$\begin{aligned}
&W^{xb}_{cd} W^{de}_{bf} W^{fg}_{ch} W^{h}_{egy} = \quad (87) \\
&\quad \frac{1}{4} W^x_{ebd} W^{bdc}_f W^{ef}_{gh} W^{gh}_{cy} + W^{xb}_{cd} W^{de}_{bf} W^{hf}_g W^{gc}_{hy} - \\
&\quad - \frac{1}{2} (W^{xe}_{bd} W^{bd}_{cf} W^{ch}_g W^{g}_{ehy} + W^{xbc}_d W^d_{ebf} W^{ef}_{gh} W^{gh}_{cy}) + \\
&\quad + \left(\frac{3}{16} W^{xb}_{cd} W^{cd}_{yb} - \frac{1}{4} W^x_{bcd} W^{bcd}_y \right) W^{ij}_{kl} W^{kl}_{ij}
\end{aligned}$$

The lemma now follows by use of Theorem 2.3, 4.2, 6.1 together with Corollary 5.3 on the right-hand-side. This method requires the Binachi identity only in the final step (through Corollary 5.3). Therefore, it seems likely that it is possible to generalize Lemma 6.2 to trace-free block symmetric double 2-forms.

It is of course possible to break the chain structures in other ways, and move towards more and more general expressions. It may also be possible to derive identities for such structures, letting new results build on previous ones. We shall however not pursue this work here, since our results are sufficient for our purposes in Section 7.

7 Applications

Definition 7.1 (The Bel-Robinson tensor) *The **Bel-Robinson tensor** in n dimensions can be defined by*

$$\begin{aligned}
\mathcal{T}_{abcd} &= C_{aecf} C^e_{b\ d}{}^f + C_{aedf} C^e_{b\ c}{}^f - \frac{1}{2} g_{ab} C_{efcg} C^{ef}{}_{d\ g} - \frac{1}{2} g_{cd} C_{aefg} C^{ef}{}_{b\ g} \\
&\quad + \frac{1}{8} g_{ab} g_{cd} C_{efgh} C^{efgh} \quad (88)
\end{aligned}$$

\diamond

where C_{abcd} denotes the Weyl tensor. The Bel-Robinson tensor is the super-energy tensor for the Weyl tensor [14]. It is of interest not only in four dimensional general relativity, but has found applications in, among other places, eleven dimensional super-gravity theories [6].

It is easy to confirm, using dimensionally dependent identities, that the Bel-Robinson tensor is trace-free in four dimensions [14], and completely symmetric in, and only in, four and five dimensions [7, 14]. Hence, in four dimensions

$$\mathcal{T}_{abcd} = \mathcal{T}_{(abcd)} \quad \mathcal{T}^k{}_{kcd} = 0 \quad (89)$$

We are now ready to state one of our main results. This, has been known for some time in the special case when the metric has a Lorentzian signature. It was first derived by Debever [5] using principal null directions, and later by Penrose [13] using spinor methods. To our knowledge there does not seem to be a proof for spaces with metrics of arbitrary signature. However, using dimensionally dependent identities, we are able to go beyond this special case and give a proof for metrics of *any signature* in four dimensions.

Theorem 7.2 *In four dimensions*

$$\mathcal{T}_{abcy}\mathcal{T}^{abcx} = \frac{1}{4}\delta_y^x\mathcal{T}_{abcd}\mathcal{T}^{abcd} \quad (90)$$

◇

Proof Simple substitution yields,

$$\begin{aligned} \mathcal{T}_{abcy}\mathcal{T}^{abcx} - \frac{1}{4}\delta_y^x\mathcal{T}_{abcd}\mathcal{T}^{abcd} = & \\ & 2C^{xb}{}_{cd}C^{de}{}_{bf}C^{fg}{}_{eh}C^{hc}{}_{gy} + 2C^{xb}{}_{cd}C^{de}{}_{bf}C^{fgc}{}_hC^h{}_{egy} - \\ & -2C^{ab}{}_{cd}C^{cd}{}_{eb}C^e{}_{gh}{}^xC_a{}^{gh}{}_y - C^{xb}{}_{cd}C^{cd}{}_{eb}C^{ef}{}_{gh}C^{gh}{}_{yf} + \\ & + \frac{1}{2}C_{abcd}C^{abcd}C^{ex}{}_{gh}C^{gh}{}_{ey} + \frac{1}{4}\delta_y^x C^{ab}{}_{cd}C^{cd}{}_{eb}C^{ef}{}_{gh}C^{gh}{}_{af} - \\ & - \frac{1}{16}\delta_y^x C_{abcd}C^{abcd}C_{efgh}C^{efgh} - \\ & - \frac{1}{4}\delta_y^x \left(2C^{ab}{}_{cd}C^{de}{}_{bf}C^{fg}{}_{eh}C^{hc}{}_{ga} + 2C^{ab}{}_{cd}C^{de}{}_{bf}C^{fgc}{}_hC^h{}_{ega} - \right. \\ & \left. - 2C^{ab}{}_{cd}C^{cd}{}_{eb}C^{ef}{}_{gh}C^{gh}{}_{af} + \frac{1}{4}C_{abcd}C^{abcd}C_{efgh}C^{efgh} \right). \quad (91) \end{aligned}$$

Using Theorem 2.3 several times on the right-hand-side, this simplifies to

$$\begin{aligned} \mathcal{T}_{abcy}\mathcal{T}^{abcx} - \frac{1}{4}\delta_y^x\mathcal{T}_{abcd}\mathcal{T}^{abcd} = & \\ & 2C^{xb}{}_{cd}C^{de}{}_{bf}C^{fg}{}_{eh}C^{hc}{}_{gy} + 2C^{xb}{}_{cd}C^{de}{}_{bf}C^{fgc}{}_hC^h{}_{egy} - \\ & - \frac{1}{4}\delta_y^x \left(2C^{ab}{}_{cd}C^{de}{}_{bf}C^{fg}{}_{eh}C^{hc}{}_{ga} + 2C^{ab}{}_{cd}C^{de}{}_{bf}C^{fgc}{}_hC^h{}_{ega} \right). \quad (92) \end{aligned}$$

The first term and the third term cancel by Theorem 3.4, and the second and the fourth term cancel by Lemma 6.2, and the theorem follows. ◇

Since we have used only the algebraic properties, it is obvious that the theorem is true not just for the Weyl tensor, but any Weyl candidate.

We have noted above that \mathcal{T}_{abcd} is completely symmetric in five as well as in four dimensions. So, a natural question to arise is whether a similar identity exists in a five dimensional space or not? One can show however, that this question is answered in the negative [8].

In the next application we turn our attention to the super-energy tensor of the Riemann tensor, also known as the Bel tensor. It may be defined in complete analogy with equation (88) by exchanging C_{abcd} for R_{abcd} [14]. The Bel tensor may be decomposed into four parts,

$$\mathcal{B}_{abcd} = \mathcal{T}_{abcd} + \mathcal{Q}_{abcd} + \mathcal{G}_{abcd} + \mathcal{E}_{abcd} \quad (93)$$

where \mathcal{T}_{abcd} is the Bel-Robinson tensor. We will address \mathcal{Q}_{abcd} and \mathcal{G}_{abcd} shortly. \mathcal{E}_{abcd} is a fairly complicated expression including products of the metric and the trace-free Ricci tensor. We will not discuss this tensor further, but we note that in four dimensions, it satisfies [8] the relation

$$\mathcal{E}_{abcy} \mathcal{E}^{abcx} = \frac{1}{4} \delta_y^x \mathcal{E}_{abcd} \mathcal{E}^{abcd} \quad (94)$$

In four dimensions [4, 8],

$$\mathcal{Q}_{abcd} = \frac{R}{6} (C_{acbd} + C_{adbc}) \quad (95)$$

$$\mathcal{G}_{abcd} = \frac{R^2}{144} (4g_a(cg_d)b - g_{ab}g_{cd}) \quad (96)$$

where R denotes the Ricci scalar. They are both easily shown [8] to satisfy a relation analogous to (94).

From a physical point of view it is reasonable to group the last two terms of equation (93) together, as done by Bonilla and Senovilla [4],

$$\mathcal{B}_{abcd} = \mathcal{T}_{abcd} + \mathcal{Q}_{abcd} + \mathcal{M}_{abcd} \quad (97)$$

The structure of the spinor formulation of the Bel tensor [2] however, suggests another way of grouping the terms of equation (93),

$$\mathcal{B}_{abcd} = \mathcal{X}_{abcd} + \mathcal{E}_{abcd} \quad (98)$$

and hence,

$$\mathcal{X}_{abcd} = \mathcal{T}_{abcd} + \mathcal{Q}_{abcd} + \mathcal{G}_{abcd}. \quad (99)$$

Since \mathcal{E}_{abcd} satisfies (94), it is natural to ask whether there is a similar relation for \mathcal{X}_{abcd} or not. As our next theorem shows, this is indeed the case.

Theorem 7.3 *In four dimensions*

$$\mathcal{X}_{abcy} \mathcal{X}^{abcx} = \frac{1}{4} \delta_y^x \mathcal{X}_{abcd} \mathcal{X}^{abcd} \quad (100)$$

◇

Proof

$$\begin{aligned} \mathcal{X}_{abcy} \mathcal{X}^{abcx} &= (\mathcal{T}_{abcy} + \mathcal{Q}_{abcy} + \mathcal{G}_{abcy}) (\mathcal{T}^{abcx} + \mathcal{Q}^{abcx} + \mathcal{G}^{abcx}) = \\ &= \mathcal{T}_{abcy} \mathcal{T}^{abcx} + \mathcal{Q}_{abcy} \mathcal{Q}^{abcx} + \mathcal{G}_{abcy} \mathcal{G}^{abcx} + \\ &\quad + \mathcal{T}_{abcy} \mathcal{Q}^{abcx} + \mathcal{Q}_{abcy} \mathcal{T}^{abcx} \end{aligned} \quad (101)$$

The cross-terms with \mathcal{G}_{abcd} vanishes since \mathcal{G}_{abcd} is essentially the metric, and \mathcal{T}_{abcd} and \mathcal{Q}_{abcd} are trace-free. For the last term, we have

$$\begin{aligned}\mathcal{Q}_{abcy}\mathcal{T}^{abcx} &= \\ &= \frac{1}{3}R \left(C^{xb}_{cd}C^{de}_{bf}C^{fc}_{ey} - C^{xb}_{cd}C^{de}_{bf}C^{fc}_{ey} + \frac{1}{2}C^{xb}_{cy}C_{bdef}C^{cdef} \right).\end{aligned}\quad (102)$$

The first term is a twisted chain of the zeroth kind, and hence we may use Corollary 5.3. The second term is a chain of the first kind and Theorem 3.4 may be applied. The third term vanishes by Theorem 2.3 and the trace-free property of C_{abcd} . Hence,

$$\mathcal{Q}_{abcy}\mathcal{T}^{abcx} = \frac{1}{12}\delta_y^x R \left(-\frac{1}{2^2}C_0^{[3]ac}_{ac} - C_1^{[3]ab}_{ab} \right) = \frac{1}{4}\delta_y^x \mathcal{Q}_{abcd}\mathcal{T}^{abcd}. \quad (103)$$

A similar argumet holds for the penultimate term of (101). Therefore, by virtue of Theorem 7.2, and analogous properites of \mathcal{Q}_{abcd} and \mathcal{G}_{abcd} , we have

$$\begin{aligned}\mathcal{X}_{abcy}\mathcal{X}^{abcx} &= \\ &= \frac{1}{4}\delta_y^x (\mathcal{T}_{abcd}\mathcal{T}^{abcd} + \mathcal{Q}_{abcd}\mathcal{Q}^{abcd} + \mathcal{G}_{abcd}\mathcal{G}^{abcd} + 2\mathcal{Q}_{abcd}\mathcal{T}^{abcd}) \\ &= \frac{1}{4}\delta_y^x \mathcal{X}_{abcd}\mathcal{X}^{abcd}\end{aligned}\quad (104)$$

as desired. \diamond

It is also possible to derive a quadratic identity for the full Bel tensor. It has a slightly different structure however, and is in a sense trivial [8].

8 Summary and discussion

Our main results, besides Theorem 7.2 and Theorem 7.3, have been the chain identities

$$U_0[m]^{xb}_{yb} = \frac{1}{4}\delta_y^x U_0[m]^{ab}_{ab} \quad (105)$$

$$U_1[m]^{xb}_{by} = \frac{1}{4}\delta_y^x U_1[m]^{ab}_{ba} \quad (106)$$

$$V_{t1}[m]^{xb}_{by} = \frac{1}{4}\delta_y^x V_{t1}[m]^{ab}_{ba} \quad (107)$$

$$W_{nt0}[m]^{wx}_{yz} = \frac{1}{2^n}W_0[m]^{wx}_{yz} \quad (108)$$

where U is a trace-free (2,2)-form, V in addition enjoys block symmetry, and W is a Weyl candidate. We have also seen that the chain structures may be broken in different ways, and that it is possible to find identities for the resulting structures.

There are several possible directions to pursue with those chain-like structures. One way is to stay with four index tensors in a four dimensional space, and investigate more and more broken structures. This sort of work may be useful in the study of relations between scalar invariants of, e.g., the Weyl tensor [9, 10].

Another direction is to start working with tensors that are *not* trace-free, and examine what the identities look like for those. Such calculations are expected to be lengthy and proper computer tools will be invaluable.

Yet another direction is to increase the number of dimensions and the number of indices of the participating tensors. This prospect looks promising in that Lovelock proved [12] a set of identities of the form,

$$S^{a_1 \dots a_{m-1} x}_{b_1 \dots b_m} S^{b_1 \dots b_m}_{a_1 \dots a_{m-1} y} = \frac{1}{n} \delta_y^x S^{a_1 \dots a_m}_{b_1 \dots b_m} S^{b_1 \dots b_m}_{a_1 \dots a_m} \quad (109)$$

where S is a trace-free (m,m)-form in an $n = 2m$ dimensional space.

It therefore seems very likely that there exist chain identities in higher dimensions for tensors with more indices in direct analogy with the relations found in Section 3. However, it will be possible to form more than just two kinds of chains.

Suppose, for instance, that $n = 6$. Then we are able to form three different kinds of chains,

$$S^{x b_1 c_1}_{d_1 e_1 f_1} S^{f_1 e_1 d_1}_{d_2 e_2 f_2} S^{f_2 e_2 d_2}_{d_3 e_3 f_3} \dots S^{f_{k-1} e_{k-1} d_{k-1}}_{c_1 b_1 y} \quad (110)$$

$$S^{x b_1 c_1}_{d_1 e_1 f_1} S^{f_1 e_1 c_2}_{c_1 e_2 f_2} S^{f_2 e_2 c_3}_{c_2 e_3 f_3} \dots S^{e_{k-1} f_{k-1} d_1}_{c_{k-1} b_1 y} \quad (111)$$

$$S^{x b_1 c_1}_{d_1 e_1 f_1} S^{f_1 b_2 c_2}_{c_1 b_1 f_2} S^{f_2 b_3 c_3}_{c_2 b_2 f_3} \dots S^{f_{k-1} e_1 d_1}_{b_{k-1} c_{k-1} y} \quad (112)$$

It may even be possible to find analogies to the twisted or more broken chain structures of Sections 4 and 5; in particular if we consider tensors with extra symmetries, e.g., block symmetry or the Bianchi like property $S_{ab[cde]f} = 0$. However, such possibilities needs further investigation.

Acknowledgements

I wish to give special thanks to Prof. Brian Edgar for introducing me to dimensionally dependent tensor identities, and his proposal to investigate the quadratic identity for the Bel-Robinson tensor.

I also wish thank Tekn D. Anders Höglund for letting me use a pre-release of his tensor manipulation program “Tensign” [15]. Without it, some of the calculations presented here would have been next to impossible.

References

- [1] Bel, L., “Sur la radiation gravitationnelle.”, *C. R. Acad. Sci.* **247** (1958), 1094-1096.
- [2] Bergqvist, G., “Positivity of General Superenergy Tensors”, *Commun. Math. Phys.* **207** (1999), 467-479.
- [3] Bergqvist, G., Höglund, A., “Algebraic Rainich theory and antisymmetrisation in higher dimensions”, *Class. Quant. Grav.* **19** (2002) 3341-3356
- [4] Bonilla, M. Á. G. and Senovilla J M.M., “Some Properties of the Bel and Bel-Robinson Tensors”, *Gen. Rel. Grav.* **29** (1997), 91-116.

- [5] Debever, R., “La super-énergie en relativité générale”, *Bulletin de la Société de Belgique* (1958), 112-147.
- [6] Deser, S., “The Immortal Bel-Robinson Tensor”, *Relativity and gravitation in general. Proc. of the Spanish Rel. Meeting in Honour of 65th Birthday of Lluís Bel*, 35-43, World Scientific Publishing Co. Pte. Ltd., Singapore New Jersey London Hong Kong, 1999, ISBN 981-02-3932-7.
- [7] Edgar, B. S. and Höglund, A., “Dimensionally dependent tensor identities by double antisymmetrization”, *J. Math. Phys.* **43** (2002), 659-677.
- [8] Edgar, B. S. and Wingbrant, O., “Old and new results for superenergy tensors from dimensionally dependent tensor identities”, *to appear in J. Math. Phys.*, gr-qc/0304099.
- [9] Harvey, A., “Identities of the scalars of the four-dimensional Riemann manifold”, *J. Math. Phys.* **28** (1987), 356-361.
- [10] Jack, I., Parker, L., “Linear independence of renormalization counterterms in curved space-times of arbitrary dimensionality”, *J. Math. Phys.* **28** (1987), 1137-1139.
- [11] Lovelock, D., “The Lanczos Identity and its Generalizations”, *Atti. Accad. Naz. Lincei, Cl. Sci. Fis., Mat. Nat., Rend* **42** (1967), 187-194.
- [12] Lovelock, D., “Dimensionally dependent identities”, *Proc. Cambridge Phil. Soc.* **68** (1970), 345-350.
- [13] Penrose, R., Rindler, W., *Spinors and Space-time vol.1*, Cambridge University Press, Cambridge, 1982.
- [14] Senovilla, J. M. M., “Super-energy tensors”, *Class. Quantum Grav.* **17** (2000), 2799-2841.
- [15] Tensign, <http://www.lysator.liu.se/~andersh/tensign>